Quest for Mathematics I (E2): Exercise sheet 1 solutions

1. (a) $\left(\frac{3n}{n+3}\right)^2 = \left(\frac{3}{1+3n^{-1}}\right)^2 \to 3^2 = 9$

   (b) $\left(\sqrt{n+2}+\sqrt{n}\right)\left(\sqrt{n+1}-\sqrt{n}\right) = \frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{1+2n^{-1}}+\sqrt{1}}{\sqrt{1+n^{-1}}+\sqrt{1}} \to 1$

   (c) $5^n - 3^n = 5^n(1-(3/5)^n) \to \infty$, since $5^n \to \infty$ and $1-(3/5)^n \to 1$

   (d) On one hand, $(5^n - 3^n)^{1/n} = 5(1-(3/5)^n)^{1/n} \leq 5$. On the other, if $x \in (0,1)$ and $n \geq 1$, then $x^n \leq x$, which implies $x \leq x^{1/n}$. It follows that $(5^n-3^n)^{1/n} = 5(1-(3/5)^n)^{1/n} \geq 5(1-(3/5)^n) \to 5$. Hence, combining the two results using the sandwich theorem gives $(5^n-3^n)^{1/n} \to 5$.

   (e) $\frac{n!}{2^n} = \frac{1}{2} \times \frac{2}{2} \times \cdots \times \frac{n}{2} \geq \frac{1}{2} \times \frac{2}{2} \times \left(\frac{3}{2}\right)^{n-2} = \frac{2}{9} \times \left(\frac{3}{2}\right)^n \to \infty$

2. If $r = 1$, then the limit is 0. If $r \in (-1,1)$, then the limit is 1. If $|r| > 1$, then the limit is -1.

3. Consider $b_n = a_n - 8$. This satisfies $b_{n+1} = \frac{3}{4}b_n$. In particular, b_n is a geometric sequence with ratio in the interval $(-1,1)$. So, $b_n \to 0$. It follows that $a_n \to 8$.

4. First, observe that, since $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = f$ for some $f > 0$, the algebra of limits gives $\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{f}$. Now, dividing the given equation by F_n (the sequence is clearly increasing, and so $F_n \neq 0$) yields

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}.$$

   Taking limits in this equation thus gives that f must satisfy

$$f = 1 + \frac{1}{f}.$$

   Rearranging gives $f^2 - f - 1 = 0$, which has roots $f = \frac{1\pm\sqrt{5}}{2}$. Only $f = \frac{1+\sqrt{5}}{2}$ satisfies $f > 0$, and so this must be the solution we are looking for.